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INCOMPLETE BLOCK DESIGNS THROUGH SYMMETRICAL FACTORIALS

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SUMMARY

The object of the present paper is to exhibit a very close link between factorials, both complete and fractional, and incomplete block designs. Several series of incomplete block designs can be obtained almost trivially by utilising the properties of such links. The method essentially consists of using different sets of numbers as the level codes for different factors instead of same set of numbers as levels for all factors and then use all these levels codes as the different varieties.

Keywords : Fractional factorial; confounding; adjusted treatments; contrasts.

Introduction

Construction of incomplete block designs was linked with factorial combinations for the first time by Yates [8] while constructing lattice designs which used to be called quasifactorials. Subsequently, Raghavarao [7] pointed a link between orthogonal arrays and semiregular group-divisible partially balanced incomplete block designs. Excepting such stray work there has not been any systematic investigation to connect combinatorics of factorial designs with that of incomplete block designs. In the present paper a very simple algorithm has been provided to link the combinatorics of these two types of designs. In short the procedure is as below. Write a number, say, b of factorial combinations of k

factors at number of levels which may all be equal or otherwise. The level codes for different factors are taken as different. Now by treating (i) the different level codes, say, v in number as symbols for as many treatments and (ii) each combination as an incomplete block of size k, we get an incomplete block design with v treatments in b blocks each of size k.

By choosing different types of factorials, various types of incomplete block designs can be obtained through the above procedure. In this paper the use of two types of symmetrical factorials namely (i) factorials where no main effects and two factor interactions are confounded (ii) factorials in which some two factor interactions are confounded have been discussed.

2. Use of Symmetrical Factorial where no Main Effects and Two Factor Interactions are Confounded

Instead of coding the levels of the factors in symmetrical factorials of the series s^n by codes 0, 1, 2, etc. for each factor, the levels of different factors can be coded differently as $(i-1)s+j$ for the ith factor after the factors are arranged in some order where j varies from 0 to $(s-1)$. The number of level codes is thus ns. Let the number of combinations in the factorial which may be complete or fractional such that no main effects and two factor interactions are confounded be b. Using the link discussed in the previous section, we get an incomplete block design with $v = sn$, $b = b$, $k = n$, $r = bk/v$. It can be shown that all such designs will be semiregular group-divisible designs.

These designs can be analysed very conveniently without following the method of Bose [2] and others. Actually the estimate of any treatment effect t_{ij} comes out as below :

$$t_{ij} = \frac{n}{(n-1)r} Q_{ij} - \frac{1}{r\,s(n-1)} S_i(Q) \tag{1}$$

where t_{ij} is the treatment corresponding to the jth level of ith factor, Q_{ij} is the adjusted total of the same treatment and $S_i(Q) = \sum_j Q_{ij}$. The adjusted treatment sum of squares due to treatments can now be obtained from $\Sigma\, t_{ij} Q_{ij}$.

It can be easily shown that

$$\text{Var}\,(t_{ij} - t_{ij'}) = \frac{2n}{(n-1)\,r}\,\sigma^2$$

$$\text{and Var}\,(t_{ij} - t_{mj'}) = \frac{2\,\sigma^2}{(n-1)\,r}\left(n - \frac{1}{s}\right) \quad (m \neq i)$$

3. Use of Factorials where some Two Factor Interactions are Confounded

Let there be $(p_i - 1)$ two factor interactions involving A_i, which are confounded in the above fractional factorial. Let the p_i factors be A_i, A_m ($m \neq i = 1, 2, \ldots, p_i$). It will be assumed that the fractional factorial containing b combinations is obtained by the method of fractionation satisfying the criteria put forth by Finney [5], Bose [2] and Das [3], [4]. In this situation in the fractonal factorial any particular level of A_i, say, jth level will occur with only one of the levels of each factor A_m with which A_i is confounded, r times and the other levels of A_m will not occur with this jth level of A_i. Taking into account this fact, the normal equation for t_{ij} will be

$$\left(r - \frac{r}{n}\right) t_{ij} - \frac{r}{n} \sum_{m \neq i} t_{mj} - \frac{r}{n\,s} \sum_{u \neq i,\, m} S_u(t) = Q_{ij} \qquad (2)$$

where t_{mj} is the treatment coming from a level of the factor A_m such that the treatment t_{ij} occurs with it in r blocks;

$$S_u(t) = \sum_{j=1}^{s} t_{uj}\,.$$

Adding these normal equations over all the treatments coming from the levels of A_i and taking the restriction

$$\sum_{u=1}^{n} S_u(t) = 0, \text{ it is seen that } S_i(t) = S_i(Q)/r.$$

Substituting the estimate of $S_i(t)$ the normal equation becomes

$$\left(r - \frac{r}{n}\right) t_{ij} - \frac{r}{n} \sum_{m \neq i} t_{mj} - \frac{1}{n\,s} \sum_{u \neq i,\, m} S_u(Q) = Q_{ij}$$

$$\text{or} \quad r t_{ij} - \frac{r}{n} \sum_{i=1}^{p_i} t_{ij} - \frac{1}{n\,s} \sum_{u \neq i,\, m} S_u\, Q = (Q)_{ij} \qquad (3)$$

Adding over the p_i equations corresponding to the p_i treatments t_{mj}, that is one for each factor which is confounded with A_i which is also counted as an affected factor, we get

$$\left(r - \frac{r}{n}\right) \sum_{i=1}^{p_i} t_{ij} - \frac{r(p_i - 1)}{n} \sum_{i=1}^{p_i} t_{ij} = \sum_{i=1}^{p_i} Q_{ij} + \frac{P_i}{ns} \sum_{u \neq i,} S_u(Q)$$

$$\text{or} \quad \frac{r}{n}\left(n - p_i\right) \sum_{i=1}^{p_i} t_{ij} = \sum_{i=1}^{p_i} Q_{ij} + \frac{p_i}{ns} \sum_{u \neq i, m} S_u(Q) \qquad (4)$$

Substituting the value of $\sum_{i=1}^{p_i} t_{ij}$ from equation (4) in equation (3) and simplifying, we get

$$t_{ij} = \frac{1}{r}\left[Q_{ij} + \frac{\sum_{i=1}^{p_i} Q_{ij}}{(n-p_i)} + \frac{1}{s(n-p_i)} \sum_{u \neq i,\, m} S_u(Q) \right] \tag{5}$$

If A_i is involved in only one two factor interaction then $p_i = 2$. When $p_i = 1$, no two factor interaction is confounded with A_i. Equation (1) in section (2) can be obtained from equation (5) by putting $p_i = 1$, $\Sigma Q_{ij} = Q_{ij}$ and $\sum_{u \neq i,\, m} S_u(Q) = -S_i(Q)$. In this case also equation (1) also gives estimates of treatment effects coming from a level of a factor not involved in affected two factor interactions.

After the estimates of t_{ij} are obtained then the adjusted treatment S.S. is obtained as usual from $\Sigma t_{ij} Q_{ij}$.

Different variances of treatment contrasts are given below :

$$V(t_{uj} - t_{uj'}) = \frac{2n}{(n-1)r}\,\sigma^2$$

where uth factor is not involved in confounded interactions.

$$V(t_{uj} - t_{u'j'}) = \frac{2\sigma^2}{r(n-1)}\left(n - \frac{1}{s}\right)$$

where both u and u' factors are not involved in confounding.

$$V(t_{ij} - t_{uj'}) = \left[\frac{1}{r(n-p_i)}\left(n - p_i + 1 - \frac{1}{s}\right) + \frac{1}{r(n-1)}\left(n - \frac{1}{s}\right)\right]\sigma^2$$

where ith factor is involved in confounding but not the uth factor.

$$V(t_{ij} - t_{mj}) = \frac{2\sigma^2}{(n-p_i)r}(n - p_i + 1) = V(t_{ij} - t_{ij'})$$

where both the ith and mth factors constitute the confounded interaction and t_{ij} and t_{mj} do not occur in any block.

$$V(t_{ij} - t_{mj}) = \frac{2\sigma^2}{r}$$

where t_{ij} and t_{mj} occur together in a block

$$V(t_{ij} - t_{i'j}) = \left[\frac{1}{r(n-p_i)} \left(n - p_i + 1 - \frac{1}{s}\right) + \frac{1}{r(n-p_i)} \left(n - p_i' + 1 - \frac{1}{s}\right)\right] \sigma^2$$

where ith and i'th factors belong to different confounded interactions.

4. Series of Designs

Some of the series of designs which can be obtained very conveniently by using the method described above are indicated below :

4.1 When the symmetrical factorial combinations used consist of $b = s^n$ combinations from k' factors each at s levels where no main effect and two factor interactions are confounded, we get the following series of designs.

$$v = k's, \quad b = s^n, \quad k = k', \quad r = s^{n-1}$$

The complete factorials come out as particular cases of the fractional factorials used above because in complete factorial no main effect and two factor interactions are confounded. It is to be presumed that the above fractional factorial is obtained by using the method of fractionation implied in the methods of construction given by Finney [5], Bose [2] and Das [3], [4]. Actually the fractional combinations consists of the appropriate principal block obtainable by the above methods.

4.2 In addition to the above methods of obtaining the fractional factorial, there are other special methods for obtaining the fractions where no main effect and two factor interactions are confounded. These are (i) the method given by Plackett and Burman [6], where the factors are each at two levels. (ii) The method given by Addleman and Kempthorne [1] where the fraction size is $b = 2\,s^n$. From these two types, the following two series of incomplete block designs can be obtained.

$$v = 2(4n - 1), \quad b = 4n, \quad k = (4n - 1), \quad r = 2n \qquad 4.2.1$$

where n is any integer.

$$v = k's, \quad b = 2s^n, \quad k = k', \quad r = 2s^{n-1} \qquad 4.2.2$$

where the symbols have the same meaning as in the designs discussed under 4.1 above.

4.3 Fractions where some two factor interactions alongwith some higher order interactions are confounded can also be used to obtain some more series of incomplete block designs. If fractions obtainable as the key block using the method of Das [4] by adding one or two different columns over and above the maximum number of columns without confounding any main effect and two factor interactions are used, then the following series of incomplete block designs can be obtained.

$$v = k's;\quad b = s^n;\quad k' = k + 1,\ k + 2;\quad r = s^{n-1}$$

where $k = (s^n - 1)/(s - 1)$ is the maximum number of factors (columns) that can be accommodated in blocks of size s^n without confounding any main effect and two factor interactions.

5. Illustration

Consider the fractional factorial $1/3^4$ (3^6). This design can not be obtained without confounding two factor interactions. The fraction is obtained by following the method of Das [4]. Here only the principal block of one replication is required. Let the 6 factors be denoted by *A*, *B*, *C*, *D*, *E* and *F*. The levels of each factor are denoted by 0, 1, and 2. The fraction obtained is shown below :

A	*B*	*C*	*D*	*E*	*F*
0	0	0	0	0	0
1	0	1	1	2	2
0	1	1	2	1	2
1	1	2	0	0	1
1	2	0	2	1	0
2	0	2	2	1	1
0	2	2	1	2	1
2	2	1	0	0	2
2	1	0	1	2	0

Confounded two factor interactions are *CF* and *DE*. Adding 3, 6, 9, 12 and 15 to each element of columns 2, 3, 4, 5 and 6 respectively, we get the following incomplete block designs:

0	3	6	9	12	15
1	3	7	10	14	17
0	4	7	11	13	17
1	4	8	9	12	16
1	5	6	11	13	15
2	3	8	11	13	16
0	5	8	10	14	16
2	5	7	9	12	17
2	4	6	10	14	15

The parameters of the designs are $v = 18$, $b = 9$, $k = 6$, $r = 3$.

The normal equations for estimating the treatment effects t_0 which comes from levels of factor A, not involved in confounded two factor interactions is given below :

$$(3 - 3/6)\ t_0 - [t_3 + t_4 + t_5 + \ldots + t_{17}]/6 = Q_0$$

or $$(5\ t_0/6) - [S_2(t) + S_3(t) + S_4(t) + S_5(t) + S_6(t)]/6 = Q_0 \qquad (6)$$

where $t_0 + t_1 + t_2 = S_1(t)$, $t_3 + t_4 + t_5 = S_2(t)$, etc.

Taking the restriction $\Sigma\ S_i(t) = 0$ and adding over similar equations for t_1 and t_2, we get

$$S_1(t) = S_1(Q)/3 \quad \text{where } S_1(Q) = Q_0 + Q_1 + Q_2$$

Substituting the value of $S_1(t)$ in equation (6), we get

$$t_0 = 2Q_0/5 - S_1(Q)/45$$

Similarly, t_1, t_2, t_3, t_4 and t_5 can be estimated as

$$t_i = 2Q_i/5 - S_1(Q)/45 \quad (i = 0, 1, 2)$$

and $$t_m = 2Q_m/5 - S_2(Q)/45 \quad (m = 3, 4, 5)$$

These estimates are evidently obtainable from formula given in Section (2) (Equation 1).

The normal equations for estimating treatment effects which comes from levels of factors involved in confounded two factor interactions are shown below for some typical cases.

$$(3 - 3/6)\, t_6 - [(t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_9 + t_{10} + t_{11} + t_{12} + t_{13} + t_{14})/6] - (3\, t_{15}/6) = Q_6$$

or $\quad 5\, t_6/2 - t_{15}/2 - [S_1(t) + S_2(t) + S_4(t) + S_5(t)]/6 = Q_6$

and $5\, t_{15}/2 - t_6/2 - [S_1(t) + S_2(t) + S_4(t) + S_5(t)]/6 = Q_{15}$

Similarly,

$$5t_9/2 - t_{12}/2 - [S_1(t) + S_2(t) + S_3(t) + S_6(t)]/6 = Q_9$$

and $5t_{12}/2 - t_9/2 - [S_1(t) + S_2(t) + S_3(t) + S_6(t)]/6 = Q_{12}$

Solving these equations as indicated in Section 3, we get

$$t_n = \frac{Q_n}{3} + \frac{Q_n + Q_{n'}}{12} - \frac{[S_i(Q) + S_{i'}(Q)]}{36}$$

where Q_n and $Q_{n'}$ belongs to two treatments from the two factors in an affected interaction and they always occur together in a block; and $S_i(Q)$ and $S_{i'}(Q)$ denotes the sum of adjusted total for the treatments from the two factors in an affected two factors interactions.

Variance of different treatment contrasts are given below :

$$\text{Var}\,(t_0 - t_1) = 4\,\sigma^2/5$$
$$\text{Var}\,(t_0 - t_3) = 34\,\sigma^2/45$$
$$\text{Var}\,(t_0 - t_6) = 23\,\sigma^2/30$$
$$\text{Var}\,(t_6 - t_7) = 5\,\sigma^2/6 = \text{Var}\,(t_6 - t_{16})$$
$$\text{Var}\,(t_6 - t_{15}) = 2\,\sigma^2/3$$
$$\text{Var}\,(t_6 - t_9) = 7\,\sigma^2/9$$

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